

STATIC FOUR-DIMENSIONAL ABELIAN BLACK HOLES IN KALUZA-KLEIN THEORY

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Abstract

Static, four-dimensional (4-d) black holes (BH's) in $(4+n)$ -d Kaluza-Klein (KK) theory with Abelian isometry and diagonal internal metric have at most one electric (Q) and one magnetic (P) charges, which can either come from the same $U(1)$ -gauge field (corresponding to BH's in effective 5-d KK theory) or from different ones (corresponding to BH's with $U(1)_M \times U(1)_E$ isometry of an effective 6-d KK theory). In the latter case, explicit non-extreme solutions have the global space-time of Schwarzschild BH's, finite temperature, and non-zero entropy. In the extreme (supersymmetric) limit the singularity becomes null, the temperature saturates the upper bound $T_H = 1/4\pi\sqrt{|QP|}$, and entropy is zero. A class of KK BH's with constrained charge configurations, exhibiting a continuous electric-magnetic duality, are generated by global $SO(n)$ transformations on the above classes of the solutions.

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I. INTRODUCTION

Kaluza-Klein (KK) compactification [1]¹ in its original sense is the procedure by which one unifies the pure 4-dimensional (4-d) gravity with gauge theories through the dimensional reduction of the higher dimensional pure gravity theories. The basic idea is that the isometry symmetry of the internal space manifests itself as the gauge symmetry in 4-d after the compactification of the extra space. In addition to gauge fields, associated with the isometry group, the scalar fields, corresponding to the degrees of freedom of the internal metric, arise upon compactification.

A class of interesting solutions for such an effective 4-d KK theory constitutes configurations with a non-trivial 4-d space-time dependence for the additional scalar fields, gauge fields, as well as for the 4-d space-time metric². In particular, spherically symmetric charged configurations correspond to charged black hole (BH) solutions with additional scalar fields varying with the spatial radial coordinate. We shall refer to such configurations as KK BH's.

Examples of KK BH's in 5-d [5–7] KK theories with Abelian isometry have been studied in the past.³ Dobiasch and Maison [6] developed a formalism by which one can generate the most general stationary, spherically symmetric solutions in Abelian KK theories. Their formalism was applied primarily to obtain the explicit solutions for the BH's in 5-d KK theory, only.

Among a class of non-trivial configurations, those which saturate the Bogomol'nyi bound on their energy (ADM mass [9]) can be viewed as non-trivial vacuum configurations – solitons – and are thus of special interest. When embedded in supersymmetric theories such configurations correspond to bosonic configurations which are invariant under (constrained) supersymmetry transformations, *i.e.*, they satisfy the corresponding Killing spinor equations. One refers to such configurations as supersymmetric ones.

Supersymmetric KK BH's in 5-d KK theory were first discussed by Gibbons and Perry [10]. Their result was recently generalized [11] to supersymmetric KK BH's in $(4 + n)$ -d Abelian KK theory, *i.e.*, those with the $U(1)^n$ internal isometry group. With a diagonal internal metric, such solutions exist if and only if the isometry group of the internal space is broken down to the $U(1)_M \times U(1)_E$ group.

The aim of this paper is to address static 4-d charged BH solutions, compatible with the corresponding Bogomol'nyi bound, in $(4 + n)$ -d KK theory with Abelian isometry group $U(1)^n$. Explicit solutions with diagonal internal metric turn out to correspond to configurations with at most one electric and one magnetic charges, which can either come from the same $U(1)$ -gauge field (corresponding to dyonic BH's in effective 5-d KK theory) or from

¹For a recent review see Ref. [2].

²One can view [4] this class of configurations as a subset of black holes in the heterotic string theory [3] compactified on a six torus. Such configurations, therefore, constitute a class of non-trivial solutions in low-energy 4-d string theory.

³See also Ref. [8].

different ones (corresponding to BH's with the $U(1)_M \times U(1)_E$ isometry in an effective 6-d KK theory). Their global space-time and thermal properties are explored. A class of Abelian KK BH solutions with constrained charge configurations is further obtained by performing global $SO(n)$ transformations on the above solutions.

The paper is organized in the following way. In Section II we discuss dimensional reduction of the $(4+n)$ -d gravity. In Section III we derive the constraints on charges for charged KK BH's with the diagonal internal metric Ansatz. In Section IV we discuss the explicit form of the non-extreme $U(1)_M \times U(1)_E$ solutions, their global space-time structure and thermal properties. In Section V we discuss the electric-magnetic duality of a more general class of the solutions generated by global $SO(n)$ rotations on the internal metric. Conclusions are given in Section VI.

II. DIMENSIONAL REDUCTION OF $(4+N)$ -DIMENSIONAL GRAVITY

The starting point is the pure Einstein-Poincaré gravity in $(4+n)$ -d:

$$\mathcal{L} = -\frac{1}{2\kappa^2} \sqrt{-g^{(4+n)}} \mathcal{R}^{(4+n)} , \quad (1)$$

where the Ricci scalar $\mathcal{R}^{(4+n)}$ and the determinant $g^{(4+n)}$ are defined in terms of a $(4+n)$ -d metric $g_{\Lambda\Pi}^{(4+n)}$, and κ is the $(4+n)$ -d gravitational constant. For the notation of space-time indices, we shall follow the convention of Ref. [11], *i.e.*, the upper- (or lower-) case letters are for those running over $(4+n)$ -d (or 4-d) space-time. Those with tilde are reserved for the n extra spatial dimensions. Latin (or greek) letters denote flat (or curved) indices. We shall use the mostly positive signature convention $(-+++)$ for the $(4+n)$ -d metric.

The dimensional reduction of the higher dimensional gravity is achieved by splitting the $(4+n)$ -d metric $g_{\Lambda\Pi}^{(4+n)}$ into the following form:

$$g_{\Lambda\Pi}^{(4+n)} = \begin{bmatrix} e^{-\frac{1}{\alpha}\varphi} g_{\lambda\pi} + e^{\frac{2\varphi}{n\alpha}} \rho_{\tilde{\lambda}\tilde{\pi}} A_{\tilde{\lambda}}^{\tilde{\lambda}} A_{\tilde{\pi}}^{\tilde{\pi}} & e^{\frac{2\varphi}{n\alpha}} \rho_{\tilde{\lambda}\tilde{\pi}} A_{\tilde{\lambda}}^{\tilde{\lambda}} \\ e^{\frac{2\varphi}{n\alpha}} \rho_{\tilde{\lambda}\tilde{\pi}} A_{\tilde{\lambda}}^{\tilde{\lambda}} & e^{\frac{2\varphi}{n\alpha}} \rho_{\tilde{\lambda}\tilde{\pi}} \end{bmatrix} , \quad (2)$$

where $\rho_{\tilde{\lambda}\tilde{\pi}}$ is the unimodular part, *i.e.*, $\det \rho_{\tilde{\lambda}\tilde{\pi}} = 1$, of the internal metric, φ is the determinant of the internal metric, which we will refer to as dilaton, and $\alpha = \sqrt{\frac{n+2}{n}}$.

The effective theory in 4-d is then obtained by imposing “the right invariance” [12] of the $(4+n)$ -d metric $g_{\Lambda\Pi}$ under the action of an isometry of the internal space. This requirement fixes the dependence of the metric components on the internal coordinates. It turns out that the internal coordinate dependence of the transformation laws (under the general coordinate transformations) of the fields in (2) factors out, and $(4+n)$ -d Einstein Lagrangian density (1) becomes independent of the internal coordinates. Then, 4-d effective action after a trivial integration over the internal space assumes the form (see for example Eq.(8) of Ref. [12]):

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} \sqrt{-g} [\mathcal{R} + e^{-\alpha\varphi} \mathcal{R}_K + \frac{1}{4} e^{\alpha\varphi} \rho_{ij} F_{\mu\nu}^i F^{j\mu\nu} + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi \\ & + \frac{1}{4} \rho^{ij} \rho^{k\ell} (D_\mu \rho_{ik}) (D^\mu \rho_{j\ell}) + \chi (\det \rho_{ij} - 1)] , \end{aligned} \quad (3)$$

where the Ricci scalar \mathcal{R}_K is defined in terms of the unimodular part ρ_{ij} ($i, j = 1, 2, \dots, n$)⁴ of the internal metric, $F_{\mu\nu}^i \equiv \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - g f_{jk}^i A_\mu^j A_\nu^k$, where f_{jk}^i is the structure constant for the internal isometry group and g is the gauge coupling constant of the isometry group, is the field strength of the gauge field A_μ^i , $D_\mu \rho_{ij} = \partial_\mu \rho_{ij} - f_{kj}^\ell A_\mu^k \rho_{i\ell}$ is the corresponding gauge covariant derivative and χ is the Lagrangian multiplier. The 4-d gravitational constant κ_4 has been set equal to 1. When the isometry group is Abelian, *i.e.*, one compactifies on n -torus T^n , the structure constant f_{jk}^i vanishes. In this case, the gauge covariant derivatives in (3) become ordinary ones and the Ricci scalar \mathcal{R}_K , which in general describes the self-interactions among scalar fields, vanishes.

III. DIAGONAL INTERNAL METRIC ANSATZ AND CONSTRAINTS ON SOLUTIONS

We shall first address the spherically symmetric configurations with the diagonal internal metric Ansatz. Then, the unimodular part of the internal metric is of the form:

$$\rho_{ij} = \text{diag}(\rho_1, \dots, \rho_{n-1}, \prod_{k=1}^{n-1} \rho_k^{-1}) \quad , \quad (4)$$

the spherically symmetric *Ansatz* for the 4-d space-time metric is of the form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\lambda(r) dt^2 + \lambda^{-1}(r) dr^2 + R(r) (d\theta^2 + \sin^2 \theta d\phi^2) \quad (5)$$

and the scalar fields φ and ρ_i depend on the radial coordinate r , only. The electromagnetic vector potentials take the form:

$$A_\phi^i = P_i \cos \theta \quad A_t^i = \psi_i(r) \quad , \quad (6)$$

where $E_i(r) = -\partial_r \psi_i(r) = \frac{\tilde{Q}_i}{R e^{\alpha\varphi} \rho_i}$ ⁵.

The *Ansatz* with all the off-diagonal components of the internal metric turned off has to be consistent with the equations of motion. In fact, such a consistency restricts the allowed charge configurations. Namely, the Euler-Lagrange equations for the components ρ_{ij} of the unimodular part of the metric is given by:

$$\frac{1}{2} e^{\alpha\varphi(r)} [R(r) E_i(r) E_j(r) - R^{-1}(r) P_i P_j] + \chi R(r) \rho^{ij}(r) = \frac{1}{2} \frac{d}{dr} [\lambda(r) R(r) \frac{d\rho^{ij}(r)}{dr}] \quad , \quad (7)$$

which implies that for the diagonal metric *Ansatz* (4) the following constraints have to be satisfied:

⁴From now on, for the simplicity of notation, we shall denote the internal space indices as $i \equiv \tilde{\alpha} - 3$, *etc.*

⁵The physical Electric charge Q_i defined in terms of the asymptotic behavior $E_i \sim \frac{Q_i}{r^2}$ ($r \rightarrow \infty$) of the electric field is related to \tilde{Q}_i through $\tilde{Q}_i = e^{\alpha\varphi_\infty} \rho_{i\infty} Q_i$. Here, φ_∞ and ρ_∞ are the constant values of the corresponding scalar fields at $r \rightarrow \infty$.

$$Q_i Q_j - e^{2\alpha\varphi} \rho_i \rho_j P_i P_j = 0 \quad i \neq j \quad . \quad (8)$$

Eqs. (8) can be satisfied if and only if ⁶

$$Q_i Q_j = 0 \quad \text{and} \quad P_i P_j = 0, \quad i \neq j \quad . \quad (9)$$

Constraints (9) imply that the same type of charge, *i.e.*, either electric or magnetic one, can appear in at most one gauge field. Consequently, the internal isometry group $U(1)^n$ is broken down to at most $U(1) \times U(1)$, with one electric and one magnetic charges, only.

When, say, the first $(n-2)$ gauge fields are turned off the first $(n-2)$ components of the diagonal internal metric become constant ($e^{\frac{2\varphi}{n\alpha}} \rho_i = \text{const.}$ ($i = 1, \dots, n-2$)). Thus, the KK BH's are those of effective 6-d KK:

$$\begin{aligned} \tilde{\mathcal{L}} = & -\frac{1}{2}\sqrt{-g}[\mathcal{R} + \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2}\partial_\mu \chi_{n-1} \partial^\mu \chi_{n-1} + \frac{1}{2}\partial_\mu \chi_n \partial^\mu \chi_n \\ & + \frac{1}{4}e^{\sqrt{2}(\Phi+\chi_{n-1})} F_{\mu\nu}^{n-1} F^{n-1 \mu\nu} + \frac{1}{4}e^{\sqrt{2}(\Phi+\chi_n)} F_{\mu\nu}^n F^{n \mu\nu}] \quad , \end{aligned} \quad (10)$$

where $\Phi \equiv \frac{\sqrt{2}}{\alpha}\varphi$ and $\chi_i \equiv \frac{1}{\sqrt{2}}[\ln \rho_i + \frac{2-n}{n\alpha}\varphi]$ ($i = n-1, n$) with the constraint $\chi_{n-1} + \chi_n = \text{const.}$

In general, one can, therefore, have the following qualitatively different classes of configurations:

- $Q_{n-1} = P_{n-1} = Q_n = P_n = 0$, which corresponds to the ordinary 4-d Schwarzschild BH's.
- Say, $Q_n = P_n = 0$, which corresponds to KK BH's in effective 5-d KK theory.
- Say, $Q_{n-1} = P_n = 0$, which corresponds to a class of KK BH's in effective 6-d KK theory, where electric and magnetic charges arise from different $U(1)$ groups.

Schwarzschild BH's are well understood, and BH's in 5-d KK theory have been extensively studied in Refs. [5–7]. We, therefore, concentrate on the study of the last class of solutions, which correspond to the non-extremal generalization of $U(1)_M \times U(1)_E$ supersymmetric solution, studied in Ref. [11].

IV. NON-EXTREME $U(1)_M \times U(1)_E$ SOLUTIONS

If only a pair of gauge fields are non-zero, the Lagrangian density (3) becomes that of effective 6-d KK theory [11]. Without loss of generality, one assumes that the first [second] gauge field to be magnetic [electric]. The Einstein's and Euler-Lagrange equations can be cast in following form:

⁶A more general constraint $Q_i Q_j - c_i c_j P_i P_j = 0$ with $c_i \equiv \rho_i e^{\alpha\varphi}$ being constant would imply the equation of motion for $\rho_i e^{\alpha\varphi}$ with $Q_i = P_i = 0$. Thus, the constraint $Q_i Q_j - c_i c_j P_i P_j = 0$ in turn reduces to the subset of constraints (9).

$$(\lambda R)'' = 2 \quad (11)$$

$$2(\lambda' R)' = R^{-1} e^{-\sqrt{2}\Phi} \rho \tilde{Q}^2 + R^{-1} e^{\sqrt{2}\Phi} \rho P^2 \quad (12)$$

$$\frac{1}{\sqrt{2}}(\lambda R \Phi')' + (\lambda R \rho^{-1} \rho')' = R^{-1} e^{\sqrt{2}\Phi} \rho P^2 \quad (13)$$

$$-\frac{1}{\sqrt{2}}(\lambda R \Phi')' + (\lambda R \rho^{-1} \rho')' = R^{-1} e^{-\sqrt{2}\Phi} \rho \tilde{Q}^2, \quad (14)$$

where $\rho \equiv e^{\sqrt{2}\chi_{n-1}}$ and the prime denotes differentiation with respect to r . Recall, $\tilde{Q} = e^{\sqrt{2}\Phi_\infty} \rho_\infty Q$. Eqs.(12)-(14) exhibit manifest electric-magnetic duality symmetry: $P \leftrightarrow \tilde{Q}$ and $\Phi \rightarrow -\Phi$.

Equation (11) implies $\lambda R = (r - r_+)(r - r_+ + 2\beta)$, while the resultant equation obtained by subtracting Eqs. (13) and (14) from Eq. (12) yields $\lambda = \frac{\rho}{\rho_\infty} \left(\frac{r - r_+}{r - r_+ + 2\beta} \right)$. Here r_+ is defined to be the outermost horizon and $\beta > 0$ is the non-extremality parameter⁷. Substitution of these relations into Eqs. (13) and (14), yields the following ordinary differential equations⁸:

$$\begin{aligned} 2\mathbf{P}^2 \frac{e^X}{(r - r_+)^2} &= [(r - r_+)(r - r_+ + 2\beta)X']' \\ 2\mathbf{Q}^2 \frac{e^Y}{(r - r_+)^2} &= [(r - r_+)(r - r_+ + 2\beta)Y']' \end{aligned} \quad (15)$$

where $X \equiv \sqrt{2}\Phi + 2\ln \lambda - \sqrt{2}\Phi_\infty$ and $Y \equiv -\sqrt{2}\Phi + 2\ln \lambda + \sqrt{2}\Phi_\infty$. Here, \mathbf{P} and \mathbf{Q} are the “screened” [11] magnetic and electric monopole charges, *i.e.*, $\mathbf{P} \equiv e^{\frac{1}{\sqrt{2}}\Phi_\infty} \rho_\infty^{\frac{1}{2}} P$ and $\mathbf{Q} \equiv e^{\frac{1}{\sqrt{2}}\Phi_\infty} \rho_\infty^{-\frac{1}{2}} Q$. The two equations in (15) can be solved explicitly, yielding the following explicit form of the solutions with regular horizons:

$$\lambda = \frac{r - r_+}{(r - r_+ + \hat{P})^{1/2} (r - r_+ + \hat{Q})^{1/2}} \quad (16)$$

$$R = r^2 \left(1 - \frac{r_+ - 2\beta}{r}\right) \left(1 - \frac{r_+ - \hat{P}}{r}\right)^{\frac{1}{2}} \left(1 - \frac{r_+ - \hat{Q}}{r}\right)^{\frac{1}{2}} \quad (17)$$

$$e^{\sqrt{2}(\Phi - \Phi_\infty)} = \frac{r - r_+ + \hat{Q}}{r - r_+ + \hat{P}} \quad (18)$$

⁷Note, the role of the non-extremality parameter β is very similar to the one used in describing non-extreme supergravity walls [13,14].

⁸These equations are reminiscent of equations for Toda molecule.

$$\rho = \rho_\infty \frac{r - r_+ + 2\beta}{(r - r_+ + \hat{P})^{1/2}(r - r_+ + \hat{Q})^{1/2}} \quad , \quad (19)$$

where $r_+ = \beta + \frac{|\mathbf{P}|\sqrt{\mathbf{P}^2 + \beta^2} - |\mathbf{Q}|\sqrt{\mathbf{Q}^2 + \beta^2}}{|\mathbf{P}| - |\mathbf{Q}|}$, $\hat{P} = \beta + \sqrt{\mathbf{P}^2 + \beta^2}$ and $\hat{Q} = \beta + \sqrt{\mathbf{Q}^2 + \beta^2}$, while the ADM mass of the configurations is of the form:

$$M = 2\beta + \sqrt{\mathbf{P}^2 + \beta^2} + \sqrt{\mathbf{Q}^2 + \beta^2} \quad . \quad (20)$$

In the limit $\beta \rightarrow 0$, the above expressions reduce to those with $r_+ = r_H = |\mathbf{P}| + |\mathbf{Q}|$, $\hat{P} = |\mathbf{P}|$, $\hat{Q} = |\mathbf{Q}|$ and the ADM mass saturates the Bogomol'nyi bound $M_{ext} = |\mathbf{P}| + |\mathbf{Q}|$. These are supersymmetric solutions of Ref. [11].

Now, we shall discuss the global space-time structure and thermal properties of the solution. We first describe the singularity structure of 4-d space-time defined by the metric coefficients (16) and (17). For the non-extreme solutions, *i.e.*, $\beta > 0$, there is a space-like singularity at $r = r_+ - 2\beta$ which is hidden behind a horizon at $r = r_+$. The global space-time of the non-extreme solutions is that of the Schwarzschild BH (see Fig. 1). In the supersymmetric (extreme) limit ($\beta \rightarrow 0$) the singularity becomes null, *i.e.*, it coincides with the horizon (see Fig. 2a). In the extreme limit with either Q or P zero, *i.e.*, supersymmetric 5-d KK BH [10], the singularity becomes naked (see Fig. 2b).

The Hawking temperature T_H , which can be calculated by identifying the inverse of the imaginary time period [16] of a functional path integral, turns out to be $T_H = \frac{|\lambda'(r_+)|}{4\pi}$. With the explicit solution (16) for λ one obtains:

$$T_H = \frac{1}{4\pi[\beta + (\mathbf{Q}^2 + \beta^2)^{\frac{1}{2}}]^{\frac{1}{2}}[\beta + (\mathbf{P}^2 + \beta^2)^{\frac{1}{2}}]^{\frac{1}{2}}} \quad . \quad (21)$$

As β decreases T_H increases and reaches the upper bound $T_{H_{ext}} = 1/4\pi|\mathbf{PQ}|$ in the extreme limit $\beta = 0$. In the extreme limit, when either \mathbf{Q} or \mathbf{P} becomes zero T_H is infinite.

The entropy S of the system, determined as $S = \frac{1}{4} \times$ (the surface area of the event horizon) [17], is of the following form:

$$S = 2\pi\beta[\beta + (\mathbf{Q}^2 + \beta^2)^{\frac{1}{2}}]^{\frac{1}{2}}[\beta + (\mathbf{P}^2 + \beta^2)^{\frac{1}{2}}]^{\frac{1}{2}} \quad . \quad (22)$$

where we have used the explicit solution (17) for $R(r)$. In the extreme limit the entropy becomes zero.

V. ELECTRIC-MAGNETIC DUALITY TRANSFORMATIONS

In addition to the 4-d general coordinate invariance and gauge symmetry, which arise from the $(4 + n)$ -d general coordinate invariance of the $(4 + n)$ -d Einstein action (1), the Lagrangian density (3) has a global $SO(n)$ invariance:

$$\rho_{ij} \rightarrow U_{ik}\rho_{kl}(U^T)_{lj} \quad A_\mu^i \rightarrow U_{ij}A_\mu^j \quad , \quad (23)$$

where U is an $SO(n)$ rotation matrix. One convenient parameterization of U is in terms of successive rotations in all the possible planes in \mathfrak{R}^n :

$$U = \hat{U}_{12} \cdots \hat{U}_{1n} \hat{U}_{23} \cdots \hat{U}_{2n} \cdots \hat{U}_{i,i+1} \cdots \hat{U}_{in} \cdots \hat{U}_{n-1,n} , \quad (24)$$

where $\hat{U}_{k\ell}$ is the rotation matrix in the (k, ℓ) -plane with a rotational angle $\theta_{k\ell}$.

The two types of solutions are obtained by performing the $SO(n)$ transformation on BH solutions of the effective 5-d and 6-d KK theories, respectively. Since the 4-d space-time metric and the dilaton field are not affected by the $SO(n)$ transformations, the global space-time and the thermal properties in each class of the solutions remain the same. The above transformations also generate non-diagonal internal metric coefficients; the $n(n-1)/2$ degrees of freedom associated with the rotational angles of the $SO(n)$ matrix generate the $n(n-1)/2$ off-diagonal internal metric components.

On the other hand, the two classes of the solutions correspond to KK BH's with constrained charge configurations, and therefore do not constitute the most general set of static 4-d Abelian KK BH's solutions. Namely, the subset of rotational matrices, which transform the charge configuration of the $U(1)_M \times U(1)_E$ solution to a new type of charge configurations, corresponds to the coset space $SO(n)/SO(n-2)$. This subset of transformations therefore provides $(2n-3)$ additional degrees of freedom for the (electric and magnetic) charge configuration. Thus, the resultant number of degrees of freedom for the charge configuration is $2n-1$. In fact, $2n$ (n -electric and n -magnetic) charges (\vec{Q}, \vec{P}) satisfy the constraint $\vec{Q} \cdot \vec{P} = 0$. Similarly, for solutions generated by the $SO(n)/SO(n-1)$ -transformations on the charge configuration of the effective 5-d KK solution, the resultant number of charge degrees of freedom is $n+1$.

In the following we shall concentrate on the electric-magnetic duality of solutions generated by $SO(n)$ transformations on BH's of the effective 6-d KK theory, since they involve less trivial transformations than those acting on BH's of the effective 5-d KK theory. The unimodular part of the internal metric and the set of n gauge fields transform in the following way:

$$\rho'_{ij} = \rho_k U_{ik} U_{jk} \quad A_\phi^i = U_{i(n-1)} P \cos \theta \quad A_t^i = U_{in} \psi(r) , \quad (25)$$

where the electric and magnetic charges of new gauge field A_μ^i are, therefore, given by $\tilde{Q}^i = U_{in} \tilde{Q}$ and $P^i = U_{i(n-1)} P$ ($i = 1, \dots, n$). The expression for 4-d metric components in the new configuration can be obtained from (16) - (17) by replacing \tilde{Q} and P by $\sqrt{\sum_{i=1}^n (\tilde{Q}^i)^2}$ and $\sqrt{\sum_{i=1}^n (P^i)^2}$, respectively.

The $SO(n)$ transformations generate the continuous electric-magnetic duality transformations which rotate the $U(1)_M \times U(1)_E$ configuration, *i.e.*, $P^i = \delta_{n-1}^i P$ and $\tilde{Q}^i = \delta_n^i \tilde{Q}$, to general charge configurations $P^i = U_{i(n-1)} P$ and $\tilde{Q}^i = U_{in} \tilde{Q}$, with the constraint $\sum_{i=1}^n P^i Q^i = 0$. This constraint is a consequence of $\sum_{i=1}^n U_{in} U_{i(n-1)} = 0$.

A subset of $SO(n)$ transformations corresponding to the rotation in the $(n-1, n)$ -plane, *i.e.*, $U = \hat{U}_{(n-1)n}$ (see Eq. (24)), mixes the monopole charges in the $(n-1)$ -th and n -th gauge fields and induces the corresponding off-diagonal terms in the internal metric of the effective 6-d KK theory. In particular, the discrete change of the rotation angle $\theta_{(n-1)n}$ from 0 to $\frac{\pi}{2}$ corresponds to a discrete electric-magnetic duality transformation, which interchanges the magnetic and electric monopole charges in the $(n-1)$ -th and n -th gauge fields [11].

VI. CONCLUSIONS

In this paper, we studied a class of static, spherically symmetric solutions in $(4 + n)$ -d KK theory with Abelian isometry $U(1)^n$. In particular, for a diagonal internal metric *Ansatz*, the consistency of the equations of motion imposes strong constraints on the possible charge configurations of such solutions; BH's exist only for configurations with at most one non-zero electric and one non-zero magnetic charges. The case of electric and magnetic charges coming from the same gauge field corresponds to BH's in effective 5-d KK theory. Configurations with electric and magnetic charges arising from different $U(1)$ gauge factors, *i.e.*, the isometry group is $U(1)_M \times U(1)_E$, correspond to BH's in effective 6-d KK theory.

Non-extreme BH's with $U(1)_M \times U(1)_E$ symmetry, which are compatible with the corresponding Bogomol'nyi bound, are parameterized in terms of the non-extremality parameter $\beta > 0$. They have a global space-time structure of Schwarzschild BH's, with the temperature T_H [entropy S] increasing [decreasing] as β decreases. In the extreme limit $\beta \rightarrow 0$, the solutions correspond to the supersymmetric BH's [11]. In this limit, the corresponding Bogomol'nyi bound for the ADM mass is saturated, T_H [S] reaches the upper [lower] limit $T_{H\text{ext}} = 1/(4\pi\sqrt{|PQ|})$ [$S = 0$], and the space-like singularity becomes null. Notably, the extreme limit is reached *smoothly*.

A class of solutions with non-diagonal internal metrics and general charge distributions among gauge fields are obtained by applying the global $SO(n)$ transformations on the solutions with diagonal internal metric. These solutions correspond to a subset of static 4-d Abelian KK BH's with constrained charge configurations. Since $SO(n)$ transformations act on the unimodular part of the internal metric and the gauge fields, only, the 4-d global space-time and the thermal properties of the general class of the 4-d abelian KK BH's are the same as those of the corresponding 4-d abelian KK BH's with the diagonal internal metric. This class of solutions exhibits continuous electric-magnetic duality symmetry parameterized by rotational angles of $SO(n)$ rotations.

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REFERENCES

- [1] T. Kaluza, *Sitz. Preuss. Akad. Wiss.* **K1** (1921) 966; O. Klein, *Z. Phys.* **37** (1926) 895; Y.M. Cho and P.G.O. Freund, *Phys. Rev.* **D12** (1975) 1711; Y.M. Cho and Pong Soo Jang, *Phys. Rev.* **D12** (1975) 3789; Y.M. Cho, *J. Math. Phys.* **16** (1975) 2029; T. Appelquist and A. Chodos, *Phys. Rev.* **D28** (1983) 772; J. Scherk and J.H. Schwarz, *Nucl. Phys.* **B153** (1979) 61.
- [2] M.J. Duff, NI-94-015 preprint (1994), hep-th # 9410046, talk given at The Oskar Klein Centenary Symposium, Stockholm, Sweden, September 19-21, 1994.
- [3] A. Sen, TIFR-TH-94-47 (November 1994), hep-th # 9411187 and references therein; M.J. Duff, R.R. Khuri, R. Minasian, J. Rahmfeld, *Nucl. Phys.* **B418** (1994) 195; M.J. Duff and J. Rahmfeld, CTP-TAMU-25-94 preprint (1994), hep-th # 9406105.
- [4] M. Cvetič and D. Youm, UPR-646-T preprint (1995), to appear.
- [5] D. Pollard, *J. Phys.* **A16** (1983) 565; D.J. Gross and M.J. Perry, *Nucl. Phys.* **B226** (1983) 29; R.D. Sorkin, *Phys. Rev. Lett.* **51** (1983) 87; Y.M. Cho and D.H. Park, *J. Math. Phys.* **31** (1990) 695.
- [6] D. Maison, *Gen. Rel. Grav.* **10** (1979) 717; P. Dobiash and D. Maison, *Gen. Rel. Grav.* **14** (1982) 231.
- [7] G.W. Gibbons and D.L. Wiltshire, *Ann. Phys.* **167** (1986) 201.
- [8] P. Breitenlohner, D. Maison and G. Gibbons, *Commun. Math. Phys.* **120** (1988) 295.
- [9] R. Arnowitt, S. Deser and C.W. Misner, *Phys. Rev.* **122** (1961) 997; L.F. Abbott and S. Deser, *Nucl. Phys.* **B195** (1982) 76.
- [10] G.W. Gibbons and M. J. Perry, *Nucl. Phys.* **B248** (1984) 629.
- [11] M. Cvetič and D. Youm, UPR-623-T preprint (1994), hep-th # 9409119, *Nucl. Phys. B*, in press.
- [12] Y.M. Cho, *Phys. Rev.* **D35** (1987) 2628.
- [13] M. Cvetič and H.H. Soleng, UPR-635-T preprint, hep-th # 9411170.
- [14] M. Cvetič, S. Griffies and H.H. Soleng, *Phys. Rev. Lett.* **71** (1993) 670; M. Cvetič, S. Griffies and H.H. Soleng, *Phys. Rev.* **D48** (1993) 2613.
- [15] S. Hawking, *Commun. Math. Phys.* **43** (1975) 199.
- [16] G.W. Gibbons and S.W. Hawking, *Phys. Rev.* **D15** (1977) 2752; G.W. Gibbons and M.J. Perry, *Proc. R. Soc. London* **A358** (1978) 467.
- [17] J.D. Bekenstein, *Phys. Rev.* **D7** (1973) 2333; **D9** (1974) 3292; S.W. Hawking, *Phys. Rev. Lett.* **26** (1971) 1344; J.M. Bardeen, B. Carter and S.W. Hawking, *Comm. Math. Phys.* **31** (1973) 161.

FIG. 1. The Penrose diagram (in the (r, t) plane) for non-extreme $U(1)_M \times U(1)_E$ 4-d Kaluza-Klein black holes. The space-like singularity (jagged line) at $r = r_+ - 2\beta$ (β is the non-extremality parameter) is hidden behind the horizon (dashed line) at $r = r_+$.

FIG. 2. The Penrose diagram (in the (r, t) plane) for the supersymmetric (extreme) $U(1)_M \times U(1)_E$ 4-d Kaluza-Klein black holes, *i.e.*, those with *both* Q and P charges non-zero, is given in fig. 2a. The Penrose diagram for the supersymmetric $U(1)_E$ (or $U(1)_M$) 4-d Kaluza-Klein black holes, *i.e.*, those with P or Q charge nonzero, is given in fig. 2b. Note a null singularity (jagged line) in the former case, and a naked singularity in the latter one.

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